Second order moments in linear smart material composites

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***Abstract*—Homogenization is a mean field approach for the determination of the effective properties of heterogeneous materials. Nevertheless further information about the field distribution can be obtained such as second order moments. The use of second order moments can notably improve the estimates of the macroscopic behavior in the nonlinear case. This has been studied mainly in the case of uncoupled behavior. We propose to define second order moments in the case of coupled elasto-magneto-electric behavior using homogenization tools. The results are compared to the field fluctuations obtained from a Finite Element model.**

I. Introduction

Homogenization is a modeling approach that enables to determine the effective behavior of heterogeneous materials [1], [2]. It makes use of the properties of the material constituents and of a limited statistical description of its microstructure. In most cases, and particularly for linear behavior, the determination of mean fields per phase is sufficient to perform the homogenization process. Nevertheless, further information about the field distribution may be necessary in some cases, particularly when dealing with nonlinear constitutive laws.

Information on field fluctuations can be obtained by determining second order moments. They can be estimated with homogenization tools. This point has been deeply investigated in the case of uncoupled (mechanical, electric, magnetic) behavior (see for instance [3], [4], [5]). We propose to define second order moments in the case of coupled behavior. Elasto-magneto-electric couplings are considered. The model relies on a previous homogenization model based on a field decomposition into several contributions depending on their physical origin [6].

In the first part, elasto-magneto-electric constitutive laws are briefly presented. In the second part, the determination of second order moments of the fields is derived in the case of coupled behavior. In the last part, this homogenization approach is applied to a piezoelectric composite. The results for second order moments are compared to Finite Element simulations.

II. Constitutive laws - Homogenization

A. Elasto-magneto-electric materials

The constitutive law of elasto-magneto-electric materials can be written in different ways, depending on the choice of the independent variables between $\mathbf{T}$ the stress tensor and $\mathbf{S}$ the strain tensor, between $\mathbf{B}$ the magnetic induction and $\mathbf{H}$ the magnetic field, and between $\mathbf{D}$ the electric induction and $\mathbf{E}$ the electric field. One possible choice is to regroup $\mathbf{T}$, $\mathbf{H}$ and $\mathbf{E}$ on one side (later referred to as $\mathbf{Y}$), and to regroup $\mathbf{S}$, $\mathbf{B}$ and $\mathbf{D}$ on the other side (later referred to as $\mathbf{X}$). The linear constitutive law reads:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{H} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbb{C} & {}^t\mathbb{q} & {}^t\mathbb{e} \\ \mathbb{q} & \mathbb{\nu} & {}^t\mathbb{\lambda} \\ \mathbb{e} & \mathbb{\lambda} & \mathbb{\beta} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{S} \\ \mathbf{B} \\ \mathbf{D} \end{pmatrix} \tag{1}$$

where $\mathbb{C}$ is the elastic stiffness tensor, $\mathbb{\nu}$ the magnetic reluctivity tensor, $\mathbb{\beta}$ the inverse permittivity tensor, $\mathbb{q}$ the piezomagnetic tensor, $\mathbb{e}$ the piezoelectric tensor and $\mathbb{\lambda}$ the magnetoelectric tensor. The constitutive law can be condensed into:

$$\mathbf{Y} = \mathbb{L} \cdot \mathbf{X} \tag{2}$$

B. Homogenization model

Homogenization models have been mainly developed in the framework of uncoupled behavior. A framework for the homogenization of coupled behavior has been recently proposed [6]. It is based on the decomposition of a n-phasic heterogeneous problem into n bi-phasic elementary inclusion problems [7]. The coupled behavior is accounted for through an appropriate decomposition of fields. The principle is to decompose the fields deriving from a potential ($\mathbf{S}$, $\mathbf{H}$ and $\mathbf{E}$) into several contributions, related to the physical origin of the field (Eq. 3).

$$\begin{cases} \mathbf{S} = \mathbf{S}^{\mathcal{C}} + \mathbf{S}^{\mathbb{\mu}} + \mathbf{S}^{\mathbb{\epsilon}} \\ \mathbf{H} = \mathbf{H}^{\mathcal{C}} + \mathbf{H}^{\mathbb{\mu}} + \mathbf{H}^{\mathbb{\epsilon}} \\ \mathbf{E} = \mathbf{E}^{\mathcal{C}} + \mathbf{E}^{\mathbb{\mu}} + \mathbf{E}^{\mathbb{\epsilon}} \end{cases} \tag{3}$$

For example, the total strain tensor $\mathbf{S}$ can be decomposed into an elastic strain $\mathbf{S}^{\mathcal{C}}$ caused by the stress $\mathbf{T}$, superimposed to a magnetism induced strain $\mathbf{S}^{\mathbb{\mu}}$ (also called magnetostriction strain) related to the magnetic state, and to an electricity induced strain $\mathbf{S}^{\mathbb{\epsilon}}$ (also called electrostriction strain) related to the electric state. The use of this decomposition allows the use of the uncoupled homogenization tools in order to express localization operators (linking the local fields to the macroscopic ones). This scheme enables to define the effective property tensor $\widetilde{\mathbb{L}}$:

$$\overline{\mathbf{Y}} = \langle \mathbf{Y} \rangle = \langle \mathbb{L} \cdot \mathbf{X} \rangle = \widetilde{\mathbb{L}} \cdot \langle \mathbf{X} \rangle = \widetilde{\mathbb{L}} \cdot \overline{\mathbf{X}} \tag{4}$$

where the operator $\langle . \rangle$ denotes an averaging operation over the whole volume of the material.

III. Second order moments

The effective property tensor $\widetilde{\mathbb{L}}$ of a composite material is classically defined as the link between the macroscopic fields $\overline{\mathbf{Y}}$ and $\overline{\mathbf{X}}$ (Eq. 4). But an energetic definition of $\widetilde{\mathbb{L}}$ could also be given, noting that, thanks to the proposed choice of independent variables, the quantity $\mathbf{Y} \cdot \delta\mathbf{X}$ represents the energy variation. Eq. 5 expresses the macroscopic energy in the composite as the average of the local energy over the volume.

$$\langle \mathbf{X} \cdot \mathbb{L} \cdot \mathbf{X} \rangle = \overline{\mathbf{X}} \cdot \widetilde{\mathbb{L}} \cdot \overline{\mathbf{X}} \tag{5}$$

Let now consider a small variation of the properties of the constituents, while maintaining constant the macroscopic field $\overline{\mathbf{X}}$. Eq. 5 becomes:

$$\langle (\mathbf{X} + \delta\mathbf{X}) \cdot (\mathbb{L} + \delta\mathbb{L}) \cdot (\mathbf{X} + \delta\mathbf{X}) \rangle = \overline{\mathbf{X}} \cdot \left(\widetilde{\mathbb{L}} + \delta\widetilde{\mathbb{L}} \right) \cdot \overline{\mathbf{X}} \tag{6}$$

Restraining Eq. 6 to first order terms leads to:

$$2 \langle \mathbf{X} \cdot \mathbb{L} \cdot \delta\mathbf{X} \rangle + \langle \mathbf{X} \cdot \delta\mathbb{L} \cdot \mathbf{X} \rangle = \overline{\mathbf{X}} \cdot \delta\widetilde{\mathbb{L}} \cdot \overline{\mathbf{X}} \tag{7}$$

The first term in the left side member is equal to zero. Indeed $\mathbf{X} \cdot \mathbb{L} \cdot \delta\mathbf{X}$ is equal to $\mathbf{Y} \cdot \delta\mathbf{X}$ that is the local variation of energy. The field distribution verifies the minimum energy principle, so that the corresponding macroscopic variation of energy is equal to zero. Finally Eq. 7 becomes:

$$\langle \mathbf{X} \cdot \delta\mathbb{L} \cdot \mathbf{X} \rangle = \overline{\mathbf{X}} \cdot \delta\widetilde{\mathbb{L}} \cdot \overline{\mathbf{X}} \tag{8}$$

The properties being uniform per phase, the averaging operation can be decomposed as follows:

$$\sum_{i=1}^{n} f_i \langle \mathbf{X} \cdot \delta\mathbb{L}_i \cdot \mathbf{X} \rangle_i = \overline{\mathbf{X}} \cdot \delta\widetilde{\mathbb{L}} \cdot \overline{\mathbf{X}} \tag{9}$$

where the operator $\langle . \rangle_i$ denotes an averaging operation over the sole phase i and f_i is the volumetric fraction of the phase i. Thus, the second order moments per phase $\langle \mathbf{X} \otimes \mathbf{X} \rangle_i$ are obtained by derivation of the effective property tensor $\widetilde{\mathbb{L}}$ with respect to the phase properties.

$$\langle \mathbf{X} \otimes \mathbf{X} \rangle_i = \frac{1}{f_i} \overline{\mathbf{X}} \cdot \frac{\partial \widetilde{\mathbb{L}}}{\partial \mathbb{L}_i} \cdot \overline{\mathbf{X}} \tag{10}$$

IV. Application to piezoelectric composites

In order to validate the proposed approach, the homogenization results (based on Eq. 10) are compared to the second order moments extracted from a Finite Element model. The composite structure studied in the Finite Element model is a sphere (piezolectric material 1) embedded in a cube (piezolectric material 2). Both phases are polarized along the z-axis. This composite is submitted to a macroscopic electric induction $\overline{\mathbf{D}}$ along the z-axis and the macroscopic strain $\overline{\mathbf{S}}$ is imposed to zero. Several volumetric fractions of the inclusion are studied. The homogenization model is performed using the piezoelectric material 2 as reference medium in the elementary inclusion problems (so-called Mori-Tanaka estimate).

The effective piezoelectric coefficients are plotted in Fig. 1 and the second order moments of the electric induction in Fig. 2. Homogenization and finite element results are close in both cases.

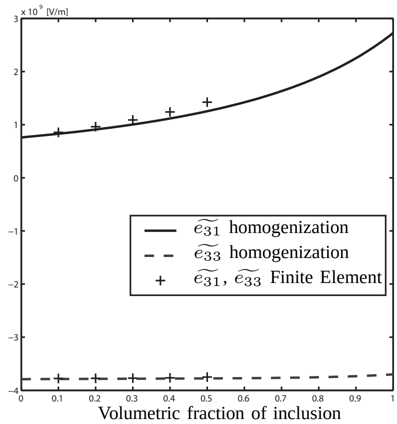

Fig. 1. Effective piezoelectric coefficients depending on the volumetric fraction of the inclusions (phase 1).

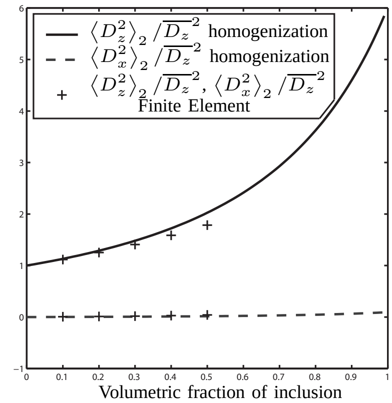

Fig. 2. Second order moments of the electric induction in phase 2 depending on the volumetric fraction of the inclusions (phase 1).

V. Conclusion

Homogenization tools have been used to determine second order moments in linear smart material composites. The main advantage of such an approach is its computational time (ratio: 10^3) compared to full field models such as Finite Element methods. The comparison to a Finite Element model for a piezoelectric composite with matrix/inclusion microstructure shows a satisfying agreement.

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